MATH 223

Some Hints and Answers for Assignment 29

Exercises 23 and 30 of Chapter 7.

23 Find the surface area of a circular cylinder of radius r and height h by rotating the graph of $f(x) = r, 0 \le x \le h$ about the x-axis.
Hint: Use the parametrization $g(t) = (t, r), 0 \le t \le h$; surface area is $2\pi h$.

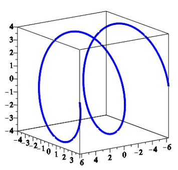
30: Sketch the solid obtained by revolving the graph of $y = 4\sqrt[3]{x}$ from (8,8) to (27, 12) around the $y-$axis and determine its surface area.
Hint: Let $g(t) = (t, 4\sqrt[3]{t}), 8 \le t \le 27$ be the parametrization. Then

$$|g'(t)| = \sqrt{1 + \frac{16}{9t^{4/3}}} = \frac{\sqrt{9t^{4/3} + 16}}{3t^{2/3}}$$

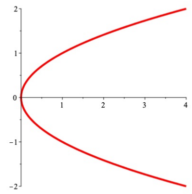
Then the surface area obtained by revolving about the **y** axis is

$$\int_8^{27} 2\pi t \frac{\sqrt{9t^{4/3} + 16}}{3t^{2/3}}\, dt = \frac{\pi}{27}\left(745^{3/2} - 160^{3/2}\right)$$

Exercise A: A curve γ has the parametrization $g(t) = (t, 4\cos t, 4\sin t)$ Sketch the curve, find its curvature and show it is constant.
Hint: We have $|g'(t)| = \sqrt{17}$. Unit tangent vector is $\mathbf{T} = \frac{1}{\sqrt{17}}(1, -4\sin t, 4\cos t)$. Curvature is $\kappa = \frac{4}{17}$.



Graph of $g(t) = (t, 4\cos t, 4\sin t), -2\pi \le t \le 2\pi$



Graph of $g(t) = (t^2, t), -2 \le t \le 2$

Exercise B: Sketch the curve with parametrization $g(t) = (t^2, t), -2 \le t \le 2$ and find its curvature at $t = 0$ and at $t = \sqrt{6}$.
Hint: . We have $g'(t) = (2t, 1)$ so

$$\mathbf{T}(t) = \left(\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}}\right) \text{ with } |\mathbf{T}'(t)| = \frac{2}{1+4t^2}.\text{Then show } \kappa(t) = \frac{2}{(1+4t^2)^{3/2}}$$

Exercise C: Suppose the curve C in the plane is the graph of the real-valued function $y = f(x)$ of one variable. Show that its curvature is

$$\frac{|f''(x)|}{(1 + |f'(x)|^2)^{3/2}}$$

Hint: To simplify the notation, we'll use F for the first derivative f' and S for the second derivative f'', simply writing f for $f(x)$, F for $f'(x)$, and S for $f''(x)$. Show

$$\mathbf{T} = \left(\frac{1}{\sqrt{1+F^2}}, \frac{F}{\sqrt{1+F^2}}\right) \text{ and } \mathbf{T}' = \left(\frac{-FS}{(1+F^2)^{3/2}}, \frac{S}{(1+F^2)^{3/2}}\right), \text{ with } |\mathbf{T}'| = \frac{|S|}{1+F^2}$$

Exercise D: *Hint:* Show that $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal to each other.

Next note that $\mathbf{T} = \frac{g'}{|g'|}$ so $g' = |g'|\mathbf{T}$. To get g'' into the picture, differentiate this last equation with respect to t using the Product Rule. Then show $g' \times g'' = g' \times |g'|\mathbf{T}' + g' \times |g'|'\mathbf{T}$

Now use $g' = |g'|\mathbf{T}$ and that $|g'|$ and $|g'|'$ are scalars to write

$$g' \times g'' = |g'|\mathbf{T} \times |g'|\mathbf{T}' + |g'|\mathbf{T} \times |g'|'\mathbf{T} = |g'||g'|\mathbf{T} \times \mathbf{T}' + |g'||g'|'\mathbf{T} \times \mathbf{T} = |g'|^2||\mathbf{T} \times \mathbf{T}'|$$